

NORMAL SUBGROUPS OF SIMPHATIC GROUPS

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ABSTRACT. A group is *SimpHAtic* if it acts geometrically on a simply connected simplicially hereditarily aspherical (*SimpHAtic*) complex. We show that finitely presented normal subgroups of *SimpHAtic* groups are either:

- finite, or
- of finite index, or
- virtually free.

This result applies, in particular, to normal subgroups of systolic groups. We prove similar strong restrictions on group extensions for other classes of asymptotically aspherical groups. The proof relies on studying homotopy types at infinity of groups in question.

In Appendix we present the topological 2-dimensional quasi-Helly property of systolic complexes.

1. INTRODUCTION

Systolic complexes were introduced for the first time by Chepoi [Che00] under the name *bridged complexes*. They were defined there as flag simplicial completions of *bridged graphs* [SC83, FJ87] – the latter appear naturally in many contexts related to metric properties of graphs and, in particular, convexity. Systolic complexes were rediscovered by Januszkiewicz-Świątkowski [JŚ06] and Haglund [Hag03], independently, in the context of geometric group theory. Namely, *systolic groups*, that is, groups acting geometrically on systolic complexes have been used to construct important new examples of highly dimensional Gromov hyperbolic groups – see e.g. [JŚ03, JŚ06, Osa13a, OŚ13].

Despite systolic groups and complexes can have arbitrarily high dimension¹, in many aspects they behave like two-dimensional objects. This phenomenon was studied thoroughly in [JŚ07, Osa07, Osa08, Świ09, OŚ13] (see also Appendix), and the intuitive picture is that systolic complexes and groups “asymptotically do not contain essential spheres”. Such behavior is

Date: January 6, 2015.

2010 Mathematics Subject Classification. 20F65, 20F67, 20F69, 57M07, 20F06.

Key words and phrases. systolic group, asymptotic asphericity, normal subgroup, topology at infinity.

Partially supported by Narodowe Centrum Nauki, decision no. DEC-2012/06/A/ST1/00259.

¹Here, we refer to various notions of *dimension*: the (virtual) cohomological one, the asymptotic one, the geometric one...

typical for objects of dimension at most two, but systolic complexes and groups, and related classes (see e.g. [CO15, Osa13b, OŚ13]) are the first highly dimensional counterparts. This implies, in particular, that systolic groups are very distinct from many classical groups, e.g. from uniform lattices in \mathbb{H}^n , for $n \geq 3$ – see [JŚ07, Osa07, Osa08, OŚ13, GO14].

In the current article we present an algebraic counterpart of the asymptotic asphericity phenomenon mentioned above. Recall that a theorem of Bieri [Bie76, Theorem B] says that a finitely presented normal subgroup of a group of cohomological dimension at most two is either free or of finite index. Surprisingly, such strong restrictions on normal subgroups can hold also in the case of objects of arbitrarily high dimension, as the two following main results show. A group is *SimpHAtic* if it acts geometrically on a simply connected simplicially hereditarily aspherical (SimpHAtic) complex (see Subsection 2.5 for the precise definition).

Theorem 1. *A finitely presented normal subgroup of a SimpHAtic group is either of finite index or virtually free.*

The following corollary is an immediate application of Theorem 1 to specific classes of SimpHAtic groups.

Corollary 1. *Finitely presented normal subgroups of groups acting geometrically on either:*

- (a) *systolic complexes, or*
- (b) *weakly systolic complexes with SD_2^* links, or*
- (c) *simply connected nonpositively curved 2-complexes, or*
- (d) *simply connected graphical small cancellation complexes*

are of finite index or virtually free.

Due to our best knowledge, even in the case of small cancellation groups the result is new.² However, the main point is that it holds also for higher dimensional groups. The case (a) establishes a conjecture by Dani Wise [Wis03, Conjecture 11.8] (originally concerning only torsion-free groups) and answers some questions from [JŚ07, Question 8.9(2)]. The case (b) deals with a subclass of weakly systolic groups studied in [Osa13b, CO15, OŚ13], containing systolic groups and exhibiting the same asphericity phenomena. In particular, the corollary extends some results from [JŚ07, Osa07, Osa13b, OŚ13]. For other classes of SimpHAtic groups to which Theorem 1 applies see e.g. [OŚ13, Examples 4.4].

Theorem 1 is a consequence of the following more general result. (π_n^∞ denotes the n -th homotopy pro-group at infinity – see Subsection 2.3.)

²Note that not all small cancellation groups fulfill the assumptions of Bieri’s theorem [Bie76], and that (d) in Corollary 1 goes beyond the case of (graphical) small cancellation groups as understood usually.

Theorem 2. *Let $n \geq 2$, and let $K \twoheadrightarrow G \twoheadrightarrow Q$ be an exact sequence of infinite groups. Assume that K and G have type F_{n+1} , $\pi_1^\infty(G) \neq 0$ and $\pi_n^\infty(G) = 0$. Then $\pi_0^\infty(K) \neq 0 \neq \pi_0^\infty(Q)$ and $\pi_{n-1}^\infty(K) = 0 = \pi_{n-1}^\infty(Q)$.*

This theorem, although quite technical in nature, is of its own interest and allows to conclude many other results around normal subgroups of groups with some asphericity properties. We present below only three consequences of Theorem 2. (The notion of asymptotic hereditary asphericity is discussed in Subsection 2.4.)

Corollary 2. *Let K be a finitely presented normal subgroup of an asymptotically hereditarily aspherical (AHA) group G of finite virtual cohomological dimension. Then K is either of finite index or virtually free.*

Theorem 2 implies also similar restrictions on quotients.

Corollary 3. *Let $K \twoheadrightarrow G \twoheadrightarrow Q$ be an exact sequence of infinite groups with finitely presented kernel K . Assume that G is a hyperbolic AHA group of finite virtual cohomological dimension. Then K is virtually non-abelian free, and Q is virtually free.*

Questions concerning AHA group extensions appear e.g. in [JS07, Question 8.9(2)] and some partial results are presented in [JS07, OŚ13]. Note that all the new examples of highly dimensional hyperbolic AHA groups constructed in [JS03, JS06, Osa13a, OŚ13] are of finite virtual cohomological dimension as in Corollary 3. It is not known in general whether hyperbolic systolic groups have finite virtual cohomological dimension. Nevertheless, the following result applies to all of them, and beyond.

Corollary 4. *Let $K \twoheadrightarrow G \twoheadrightarrow Q$ be an exact sequence of infinite groups with finitely presented kernel K . Assume that G is a SimpHATIC hyperbolic group. Then K is virtually non-abelian free, and Q is virtually free.*

Finally, let us mention that a corollary of Theorem 2 is crucial for a construction of some optimal classifying spaces for systolic groups, which will be presented in a forthcoming paper.

Ideas of the proofs. The proof of Theorem 2 relies on the analysis of the homotopy type at infinity (proper homotopy type) of the group G (denoted $\partial^\infty G$ in Figure 1), and we proceed by a contradiction. We restrict here considerations to the case $n = 2$ – this is enough for proving Theorem 1. Suppose that $\pi_1^\infty(K) \neq 0$. Since K, Q are infinite, G has one end, and furthermore K, Q are not one-ended, by $\pi_1^\infty(G) \neq 0$ (see Section 3). Therefore, by theorems of Stallings and of Dunwoody the 1-homotopy type at infinity of K, Q ($\partial^\infty K$ and $\partial^\infty Q$ in Figure 1) is the type of a disjoint union of some points and some non-simply connected pieces (see Subsections 3.1 & 3.2). Furthermore, the homotopy type at infinity of G is the join of the types of K and Q (see Subsections 3.3 & 3.4). However, such join contains nontrivial 2-spheres appearing in suspensions of non-simply connected factors coming

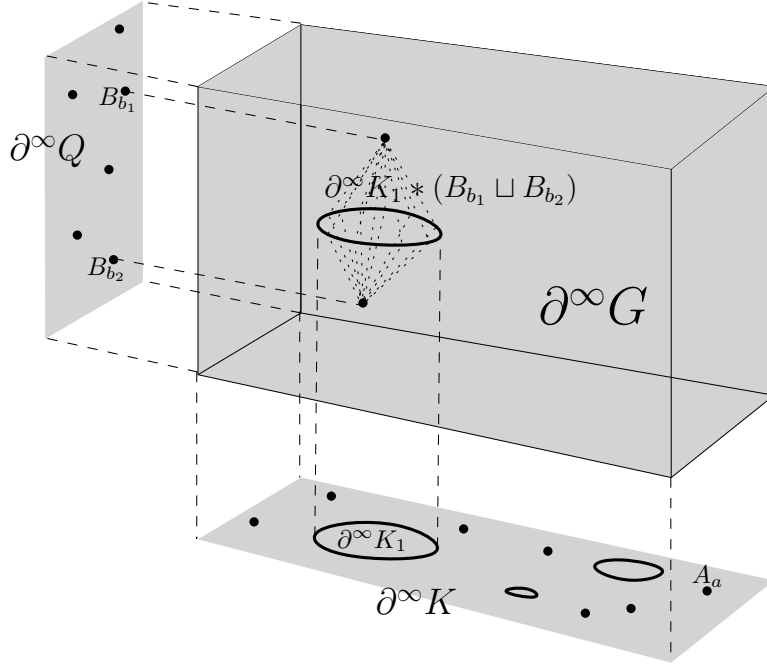


FIGURE 1. Idea of the proof of Theorem 2 in the case $n = 2$: The homotopy type at infinity $\partial^\infty G$ of G . (Some notations come from Section 3.)

from K (the join $\partial^\infty K_1 * (B_{b_1} \sqcup B_{b_2})$ in Figure 1). Therefore, $\pi_2^\infty G \neq 0$ – contradiction.

For Theorem 1, one observes that $\pi_1^\infty(K) = 0$ only when there are no one-ended factors in the splitting of K . This relies on the fact that finitely presented one-ended subgroups of a Simplicial G are not simply connected at infinity, since they are again Simplicial.

Article's structure. In the next Section 2 we present some preliminary notions and results concerning e.g.: weakly systolic complexes and groups (Subsection 2.2), proper homotopy type (Subsection 2.3), asymptotic hereditary asphericity (Subsection 2.4), and Simplicial groups (Subsection 2.5). In Section 3 we prove the main Theorem 2. In Section 4 we present the proofs of Theorem 1 and Corollary 1. In Section 5 we prove Corollaries 2, 3 & 4. Few remarks are given in Section 6.

In Appendix we present the topological 2-dimensional quasi-Helly property of systolic complexes – Theorem A.

2. PRELIMINARIES

2.1. Complexes and groups. Throughout this article we work usually (if not stated otherwise) with regular CW complexes, called further simply *complexes* (see e.g. [Geo08] for some notation). Accordingly, all *maps* are

cellular maps. We call a complex G -complex if G acts on it by cellular automorphisms. The action is *proper*, or the complex is a *proper G -complex* if stabilizers of cells are finite. An n -skeleton of a complex X is denoted by $X^{(n)}$. A G -complex X has the n -skeleton *finite mod G* if the quotient $X^{(n)}/G$ is a finite complex. Recall that a group has *type F_n* if there is a *free* (that is, with trivial stabilizers of cells) contractible G -complex with the n -skeleton finite mod G . A group G acts *geometrically* on X , if X is a finite dimensional proper contractible G -complex with every skeleton finite mod G . In particular, X is then uniformly locally finite. We usually work with the path metric on $X^{(0)}$ defined by lengths of paths in $X^{(1)}$ connecting two given vertices. It follows that if a finitely generated G acts geometrically on X , then $X^{(0)}$ is quasi-isometric to G equipped with a word metric coming from a finite generating set. For a subcomplex Y of a complex X , by $X - Y$ we denote the smallest subcomplex of X containing $X \setminus Y$ (set-theoretic difference).

2.2. Weakly systolic complexes and groups. In this subsection we restrict our studies to simplicial complexes. We follow mostly notation from e.g. [JS06, Osa07, Osa08, Osa13b, Osa13a]. A simplicial complex X is *flag* if it is determined by its 1-skeleton $X^{(1)}$, that is, if every set of vertices pairwise connected by edges spans a simplex in X . A subcomplex Y of a simplicial complex X is *full* if it is determined by its vertices, that is, if any set of vertices in Y contained in a simplex of X , span a simplex in Y . The *link* of a simplex σ of X is the simplicial complex being the union of all simplices τ of X disjoint from σ , but spanning together with σ a simplex of X . A k -cycle is a triangulation of the circle S^1 consisting of k edges and k vertices. For $k \geq 5$, a flag simplicial complex X is *k -large* (respectively, *locally k -large*) if there are no full i -cycles (as subcomplexes) in X (respectively, in any link of X), for $i < k$. A complex X is *k -systolic* (respectively, *systolic*) if it is simply connected and locally k -large (respectively, locally 6-large) – see [JS06]. A *k -wheel* is a graph consisting of a k -cycle and a vertex adjacent to all vertices of the cycle. A *k -wheel with a pendant triangle* is a graph consisting of a k -wheel and a vertex adjacent to two vertices of the k -cycle. A complex X satisfies the SD_2^* *property* if it is locally 5-large and every 5-wheel with a pendant triangle is contained in a 1-ball around some vertex of X . A flag simplicial complex is *weakly systolic* if it is simply connected and satisfies the SD_2^* property – see [Osa13b, CO15]. A weakly systolic complex in which all links satisfy the SD_2^* property is called a *weakly systolic complex with SD_2^* links*. A group is *systolic* (respectively, *k -systolic*, *weakly systolic*) if it acts geometrically on a systolic (respectively, k -systolic, weakly systolic) complex.

2.3. Proper homotopy type. The basics about proper homotopy theory can be found in [Geo08]. Recall that a cellular map $f: X \rightarrow Y$ between complexes is *proper* if $f^{-1}(C) \cap X^{(n)}$ is a finite subcomplex for any finite

subcomplex $C \subset Y$ and any n (see e.g. [Geo08, Chapter 10], where the name *CW-proper* is used). Two maps $f, f': X \rightarrow Y$ are *properly homotopic*, denoted $f \simeq_p f'$, if there exists a proper map $F: X \times [0, 1] \rightarrow Y$ with $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$, for all $x \in X$. A proper map $f: X \rightarrow Y$ is a *proper homotopy equivalence* (respectively, *proper n -equivalence*) if there exists a *proper homotopy inverse* (respectively, *proper n -inverse*) $f^{-1}: Y \rightarrow X$ with $f^{-1} \circ f \simeq_p \text{id}_X$, and $f \circ f^{-1} \simeq_p \text{id}_Y$ (respectively, $f^{-1} \circ f|_{X^{(n)}} \simeq_p \text{id}_{X^{(n)}}$, and $f \circ f^{-1}|_{Y^{(n)}} \simeq_p \text{id}_{Y^{(n)}}$). In such case we say that X and Y have the same *proper homotopy type* (respectively, *proper n -type*).

For $n \geq 0$, we say that the *n -homotopy pro-group at infinity vanishes* for X , denoted $\pi_n^\infty(X) = 0$, if the following holds. For every finite subcomplex $M \subset X$ there exists a subcomplex M' containing M such that: For any map $f: S^n \rightarrow X - M'$ of the n -sphere S^n , there exists its extension $F: B^{n+1} \rightarrow X - M$. If $\pi_n^\infty(X) = 0$ does not hold, then we write $\pi_n^\infty(X) \neq 0$. The condition $\pi_0^\infty(X) = 0$ means that X is *one-ended*, and $\pi_0^\infty(X) = \pi_1^\infty(X) = 0$ means *simple connectedness at infinity*. The next proposition shows that for a group acting in a nice way on a complex, vanishing of homotopy pro-groups at infinity does not depend on the complex. It is a version of [Osa07, Proposition 2.10] whose proof is analogous to the one in [Osa07]. (We may omit the requirement of ‘rigidity’ here, since an action induced on the subdivision of a regular CW complex is rigid.) Compare also [Geo08, Theorems 17.2.3 & 17.2.4] and [OŚ13, Theorem 7.2], and proofs there.

Proposition 3. *Let X, Y be two n -connected proper G -complexes with $(n+1)$ -skeleta finite mod G . Then $\pi_n^\infty(X) = 0$ iff $\pi_n^\infty(Y) = 0$.*

Therefore, we say that $\pi_n^\infty(G) = 0$ (respectively, $\pi_n^\infty(G) \neq 0$) if $\pi_n^\infty(X) = 0$ (respectively, $\pi_n^\infty(X) \neq 0$), for some (and hence, any) G -complex as in Proposition 3.

Similarly (cf. e.g. [OŚ13, Theorem 7.2]), vanishing of the homotopy pro-groups at infinity is an invariant of the coarse equivalence, and hence of quasi-isometry. However, directly from the definition above it follows that this is also an invariant of the proper homotopy type. More precisely, if there is a proper $(n+1)$ -equivalence $f: X \rightarrow Y$, then $\pi_n^\infty(X) = 0$ iff $\pi_n^\infty(Y) = 0$ – cf. [Geo08, Chapter 17]. This invariance is crucial in our approach – see Subsection 3.2.

2.4. Asymptotic hereditary asphericity (AHA). The notion of asymptotic hereditary asphericity (AHA) was introduced in [JŚ07]. We follow here mostly notations from [OŚ13]. Given an integer $i \geq 0$, for subsets $C \subseteq D$ of a metric space (X, d) we say that C is *$(n; r, R)$ -aspherical in D* if every simplicial map $f: S \rightarrow P_r(C)$ (where P_r denotes the Rips complex with constant r), where S is a triangulation of the n -sphere S^n , has a simplicial extension $F: B \rightarrow P_R(D)$, for some triangulation B of the $(n+1)$ -ball B^{n+1} .

such that $\partial B = S$. A metric space X is *asymptotically hereditarily aspherical*, shortly *AHA*, if for every $r > 0$ there exists $R > 0$ such that every subset $A \subseteq X$ is $(n; r, R)$ -aspherical in itself, for every $n \geq 2$. A finitely generated group is AHA if it is AHA as a metric space for some (and hence any) word metric coming from a finite generating set. A subgroup of an AHA group is AHA [JŚ07, Corollary 3.4]. A finitely presented AHA group G has type F_∞ [OŚ13, Theorem C], and $\pi_n^\infty(G) = 0$ for all $n \geq 2$ [OŚ13, Theorem D]. If moreover, the virtual cohomological dimension of G is finite, or if G acts geometrically on a contractible complex of finite dimension, then $\pi_1^\infty(G) \neq 0$ [OŚ13, Theorem 7.6].

For examples of AHA groups see the next subsection.

2.5. SimpHatic groups. In this subsection we introduce the notions of SimpHatic complexes and SimpHatic groups.

Definition 1. A flag simplicial complex is *simplicially hereditarily aspherical*, shortly *SimpHatic*, if every its full subcomplex is aspherical.

A group acting geometrically on a simply connected SimpHatic complex is called *SimpHatic*.

Let us note few simple facts about SimpHatic groups:

- (1) A simply connected SimpHatic complex is contractible, by Whitehead's theorem (see e.g. [Hat02, Theorem 4.5] or [Geo08, Proposition 4.1.4]).
- (2) SimpHatic groups are AHA, by [OŚ13, Corollary 3.2].
- (3) Weakly systolic complexes with SD_2^* links are SimpHatic by [Osa13b, Proposition 8.2]. In particular, systolic complexes are SimpHatic, and systolic groups are SimpHatic (see [JŚ06]).
- (4) There are SimpHatic groups that are not weakly systolic – e.g. some Baumslag-Solitar groups. For other examples of SimpHatic groups see e.g. [OŚ13].
- (5) Full subcomplexes and covers of SimpHatic complexes are again SimpHatic. Therefore, by a theorem of Hanlon-Martínez-Pedroza [HMP14, Theorem 1.1], finitely presented subgroups of SimpHatic groups are again SimpHatic.

We cannot resist to formulate here the famous Whitehead's asphericity question in the following way:

Whitehead's Problem. *Are all aspherical 2-complexes SimpHatic?*

3. PROOF OF THEOREM 2

In this section we present the proof of Theorem 2 from Introduction. Recall that $K \twoheadrightarrow G \twoheadrightarrow Q$ is an exact sequence of infinite groups. Since K, G have type F_{n+1} , the image Q has type F_{n+1} as well – see e.g. [Geo08, Theorem 7.2.21]. Since K, Q are infinite, we have that G has one end, that is, $\pi_0^\infty(G) = 0$ – see e.g. [Geo08, Corollary 16.8.5]. By the result of

Houghton [Hou77] and Jackson [Jac82] (see also [Geo08, Corollary 16.8.5]), from the fact that $\pi_1^\infty(G) \neq 0$ we conclude that neither K nor Q has one end. For the rest of the proof we suppose that $\pi_{n-1}^\infty(K) \neq 0$ and we will show that this leads to a contradiction. The other case – under the assumption $\pi_{n-1}^\infty(Q) \neq 0$ – is essentially the same. This will conclude the proof.

3.1. Decomposing K and Q . Since the kernel K has more than one end, by the result of Stallings [Sta71], it decomposes into a free product with amalgamation over a finite group, or as an HNN extension over a finite group. If the factors are not one-ended we may repeat decomposing further. A theorem of Dunwoody [Dun85] says that finitely presented groups are accessible, that is, such a procedure ends after finitely many steps. As a result, by the classical Bass-Serre theory (see e.g. [Geo08, Chapter 6.2]), the group K is the fundamental group of a finite graph of groups \mathcal{K} with vertex groups having at most one end, and finite edge groups. Let $\{K_i\}$ be the collection of the vertex groups, and let $\{L_j\}$ be the set of edge groups of \mathcal{K} . Similarly Q is the fundamental group of a graph of groups \mathcal{Q} , with vertex groups $\{Q_i\}$. Since $\pi_{n-1}^\infty(K) \neq 0$, at least one group from $\{K_i\}$, say K_1 , is one-ended and with $\pi_{n-1}^\infty(K_1) \neq 0$.

3.2. Classifying spaces for K and Q . Since K, Q have type F_{n+1} , all the factors K_i, Q_j have also type F_{n+1} . For all i , let Y_i (respectively, Z_i) be a contractible free K_i -complex (respectively, Q_i -complex) with the $(n+1)$ -skeleton finite mod K_i (respectively, Q_i). By a general technique (see e.g. [Geo08, Chapter 6.2], and [Hat02, Chapter 1.B]) one may compose all Y_i 's to obtain a tree of complexes Y being a contractible proper K -complex with the $(n+1)$ -skeleton finite mod K . Moreover, subject to modifying slightly each Y_i (by e.g. coning-off L_j -invariant subcomplexes), the complex Y may be arranged in such a way that the following condition is satisfied: Y is a union of copies of Y_i 's, and any two copies have at most one vertex in common. Similarly, there is a contractible proper Q -complex with the $(n+1)$ -skeleton finite mod Q , which is a union of copies of Z_i 's and any two copies have at most one vertex in common.

Since Y and Z have $(n+1)$ -skeleta finite mod K and Q , and since G has type F_{n+1} , there is a contractible proper G -complex X with the following properties (see e.g. [Geo08, Proposition 17.3.4]): The $(n+1)$ -skeleton of X is finite mod G , and there is a proper $(n+1)$ -equivalence between X and $Y \times Z$. Therefore, to obtain a contradiction under our assumptions, it is enough to show that $\pi_n^\infty(Y \times Z) \neq 0$: By the proper $(n+1)$ -equivalence, it implies $\pi_n^\infty(X) \neq 0$ and hence, by Proposition 3, $\pi_n^\infty(G) \neq 0$ – contradiction. In the next subsection we study the topology at infinity of $Y \times Z$.

3.3. Topology at infinity of $Y \times Z$. Our goal is to show that $\pi_n^\infty(Y \times Z) \neq 0$. To do this we need to find subcomplexes $M \subset Y$ and $N \subset Z$ with finite $(n+1)$ -skeleta, and having the following property: For any $C \subset Y \times Z$ with finite $(n+1)$ -skeleton and containing $M \times N$, there is a map $S^n \rightarrow$

$Y \times Z - C$ that is not homotopically trivial within $Y \times Z - M \times N$. In the current subsection we define M and N , and we study the homotopy type of $Y \times Z - M \times N$.

Pick a copy of Y_1 in Y , and a copy of Z_1 in Z . We further denote them by Y_1 and Z_1 , respectively – this should not lead to confusion. Since $\pi_{n-1}^\infty(K_1) \neq 0$, by Proposition 3, $\pi_{n-1}^\infty(Y_1) \neq 0$. Therefore, there exists a subcomplex $M \subset Y_1$ with finite $(n+1)$ -skeleton and such that the following holds: For every $M' \subset Y_1$ with finite $(n+1)$ -skeleton and containing M , there exists a sphere $S^{n-1} \rightarrow Y_1 - M'$ not homotopically trivial within $Y_1 - M$. Adding path-connected components of $Y_1 - M$ with finite $(n+1)$ -skeleton to M we may further assume that $Y_1 - M$ is path-connected. Similarly, we choose a subcomplex $N \subset Z_1$ with finite $(n+1)$ -skeleton and such that $Z_1 - N$ is path-connected. We claim that M, N are as required above, and in the rest of this subsection we study the homotopy type of $Y \times Z - M \times N$.

Since the $(n+1)$ -skeleton of Y (respectively, Z) is locally finite, and by our construction of Y (respectively, Z), the set $A := Y_1 \cap (Y - Y_1)$ (respectively, $B := Z_1 \cap (Z - Z_1)$) is a discrete set of vertices. Since the $(n+1)$ -skeleton of M (respectively, N) is finite, the set $A \cap M$ (respectively, $B \cap N$) is an empty or finite set $\{a_1, a_2, \dots, a_m\}$ (respectively, $\{b_1, b_2, \dots, b_s\}$). By possibly enlarging N within Z_1 we may assume that $|B \cap N| \geq 2$, that is, $s \geq 2$. For $a \in A$ (respectively, $b \in B$) let A_a (respectively, B_b) denote the closure in Y (respectively, in Z) of the connected component of $Y \setminus \{a\}$ (respectively, $Z \setminus \{b\}$), not containing Y_1 (respectively, Z_1). Observe that such closure is the union of the component and $\{a\}$ (respectively, $\{b\}$) – see Figure 2 (left).

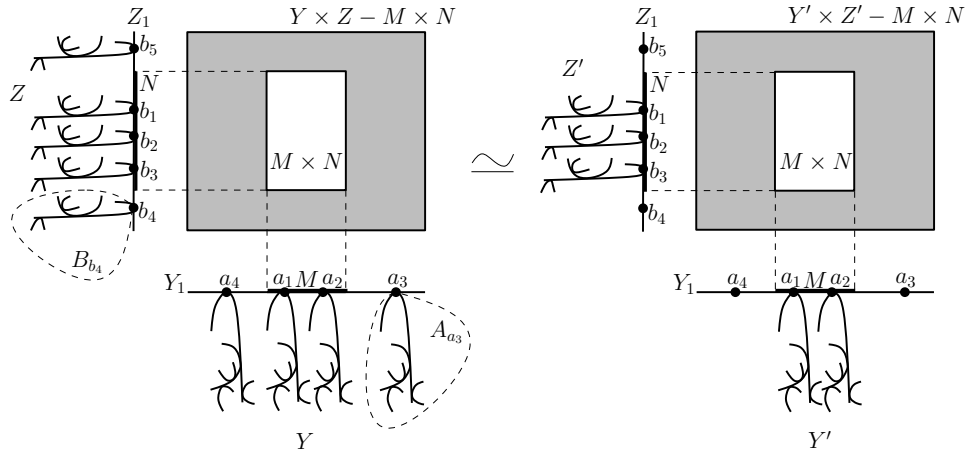


FIGURE 2. From $Y \times Z - M \times N$ to $Y' \times Z' - M \times N$.

Since A_a (respectively, B_b) is contractible, for $a \notin M$ (respectively, $b \notin N$) we may contract A_a to a (respectively, B_b to b) not changing the homotopy type of $Y \times Z - M \times N$. Therefore, let Y' (respectively, Z') denote Y

(respectively, Z) with A_a (respectively, B_b), contracted to a (respectively b), for all $a \notin M$ (respectively, $b \notin N$) – see Figure 2 (right). We have:

$$(1) \quad \begin{aligned} Y' &= Y_1 \sqcup (A_{a_1} \sqcup A_{a_2} \sqcup \dots \sqcup A_{a_m}) \\ Z' &= Z_1 \sqcup (B_{b_1} \sqcup B_{b_2} \sqcup \dots \sqcup B_{b_s}), \end{aligned}$$

and

$$(2) \quad Y \times Z - M \times N \simeq Y' \times Z' - M \times N.$$

Furthermore, by formula (1), and since $M \subset Y_1$ (respectively, $N \subset Z_1$) and $\{a_1, a_2, \dots, a_m\} \subset M$ (respectively, $\{b_1, b_2, \dots, b_s\} \subset N$), we have

$$(3) \quad \begin{aligned} Y' \times Z' - M \times N &= (Y_1 \times Z_1 - M \times N) \sqcup \\ &\quad \sqcup Y_1 \times (B_{b_1} \sqcup \dots \sqcup B_{b_s}) \sqcup \\ &\quad \sqcup (A_{a_1} \sqcup \dots \sqcup A_{a_m}) \times Z_1 \sqcup \\ &\quad \sqcup (A_{a_1} \sqcup \dots \sqcup A_{a_m}) \times (B_{b_1} \sqcup \dots \sqcup B_{b_s}), \end{aligned}$$

where $Y_1 \times Z_1 - M \times N$ and $Y_1 \times (B_{b_1} \sqcup \dots \sqcup B_{b_s})$ intersect along $(Y_1 - M) \times (\{b_1\} \sqcup \dots \sqcup \{b_s\})$, etc.

Again, we may contract subcomplexes of the type $Y_1 \times B_{b_j}$ to $Y_1 \times \{b_j\}$, $A_{a_i} \times Z_1$ to $\{a_i\} \times Z_1$, and $A_{a_i} \times B_{b_j}$ to $\{a_i\} \times \{b_j\}$, obtaining the following homotopy equivalence – see Figure 3.

$$(4) \quad \begin{aligned} Y' \times Z' - M \times N &\simeq (Y_1 \times Z_1 - M \times N) \sqcup \\ &\quad \sqcup Y_1 \times (\{b_1\} \sqcup \dots \sqcup \{b_s\}) \sqcup \\ &\quad \sqcup (\{a_1\} \sqcup \dots \sqcup \{a_m\}) \times Z_1 \sqcup \\ &\quad \sqcup (\{a_1\} \sqcup \dots \sqcup \{a_m\}) \times (\{b_1\} \sqcup \dots \sqcup \{b_s\}), \end{aligned}$$

where (see Figure 3):

$$(5) \quad \begin{aligned} (Y_1 \times Z_1 - M \times N) \cap [Y_1 \times (\{b_1\} \sqcup \dots \sqcup \{b_s\})] &= \\ &= (Y_1 - M) \times (\{b_1\} \sqcup \dots \sqcup \{b_s\}), \\ (Y_1 \times Z_1 - M \times N) \cap [(\{a_1\} \sqcup \dots \sqcup \{a_m\}) \times Z_1] &= \\ &= (\{a_1\} \sqcup \dots \sqcup \{a_m\}) \times (Z_1 - N), \\ \text{and } (\{a_i\} \times Z_1) \cap (Y_1 \times \{b_j\}) &= \{a_i\} \times \{b_j\}. \end{aligned}$$

Since Y_1, Z_1 are contractible, the subcomplexes $(Y_1 - M) \times \{b_j\}$ and $\{a_i\} \times (Z_1 - N)$ can be contracted to points within $Y_1 \times Z_1 - M \times N$. (This may be done by “sliding” $(Y_1 - M) \times \{b_j\}$ to $(Y_1 - M) \times \{x\}$, for some $x \notin N$ first, and then contracting $(Y_1 - M) \times \{x\}$ to a point within $Y_1 \times \{x\}$.) Therefore, by (4) and (5) we obtain the following homotopy equivalence – see Figure 4:

$$(6) \quad \begin{aligned} Y' \times Z' - M \times N &\simeq (Y_1 \times Z_1 - M \times N) \sqcup \\ &\quad \sqcup (Y_1 / (Y_1 - M) \sqcup \dots \sqcup Y_1 / (Y_1 - M)) \sqcup \\ &\quad \sqcup (Z_1 / (Z_1 - N) \sqcup \dots \sqcup Z_1 / (Z_1 - N)), \end{aligned}$$

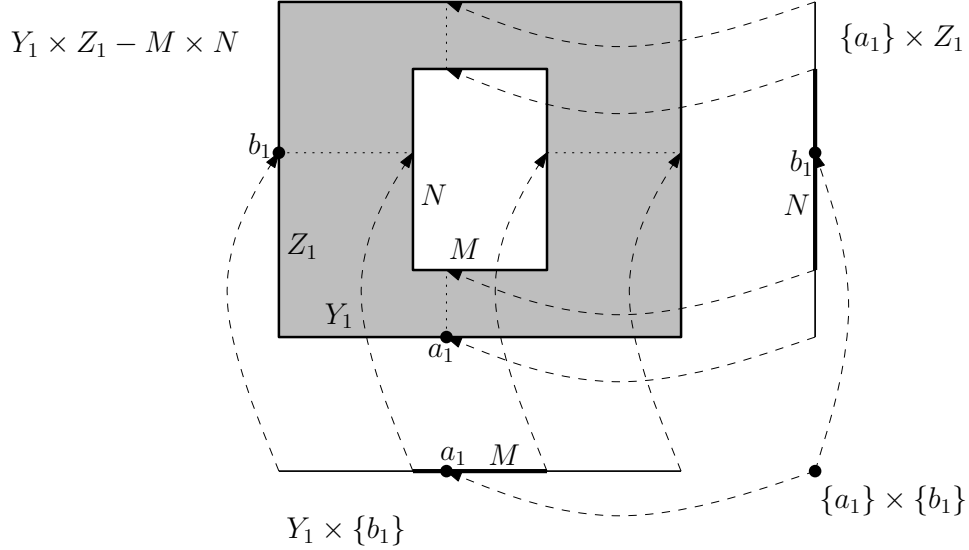


FIGURE 3. Homotopy type of $Y' \times Z' - M \times N$. (Only a_1 and b_1 showed among a_i 's and b_j 's.)

where $Y_1/(Y_1 - M)$ (respectively, $Z_1/(Z_1 - N)$) is the complex obtained from Y_1 (respectively, Z_1), by contracting the subcomplex $Y_1 - M$ (respectively, $Z_1 - N$) to a point. The number of copies of $Y_1/(Y_1 - M)$ in (6) is s , and the number of copies of $Z_1/(Z_1 - N)$ is m . For all $i = 1, \dots, m$ and $j = 1, \dots, s$, the point a_i in the j -th copy of $Y_1/(Y_1 - M)$ (that is, in the complex obtained after modifying $Y_1 \times \{b_j\}$) is identified with the point b_j in the i -th copy of $Z_1/(Z_1 - N)$ – see Figure 4. We denote by \hat{Y}_1 and \hat{Y}_2 the copies of $Y_1/(Y_1 - M)$ corresponding to, respectively, b_1 and b_2 .

Therefore, the space $Y' \times Z' - M \times N$ and thus also $Y \times Z - M \times N$ have the homotopy type

$$(7) \quad Y \times Z - M \times N \simeq (\hat{Y}_1 \vee \hat{Y}_2) \vee U$$

of the wedge of \hat{Y}_1 , \hat{Y}_2 and some union U of $Y_1 \times Z_1 - M \times N$, and (possibly) other copies of $Y_1/(Y_1 - M)$ and $Z_1/(Z_1 - N)$.

3.4. Finalizing the proof of Theorem 2. Let a subcomplex $C \subset Y \times Z$ with finite $(n + 1)$ -skeleton, and containing $M \times N$ be given. We will see that there exists an n -sphere $\varphi: S^n \rightarrow Y \times Z - C$ that is not homotopically trivial within $Y \times Z - M \times N$. Let $M' \subset Y$ and $N' \subset Z$ be subcomplexes with finite $(n + 1)$ -skeleta such that $C \subset M' \times N'$. Fix a vertex $b^0 \in Z_1$. By our choice of M , there exists a sphere $\psi: S^{n-1} \rightarrow (Y_1 - M') \times \{b^0\} \subset (Y_1 - M') \times Z_1$ that is not contractible within $Y_1 - M$. Using ψ , we will construct a sphere $\varphi: S^n \rightarrow Y \times Z - M' \times N'$. Then we will show that this sphere is homotopically nontrivial within $Y \times Z - M \times N$.

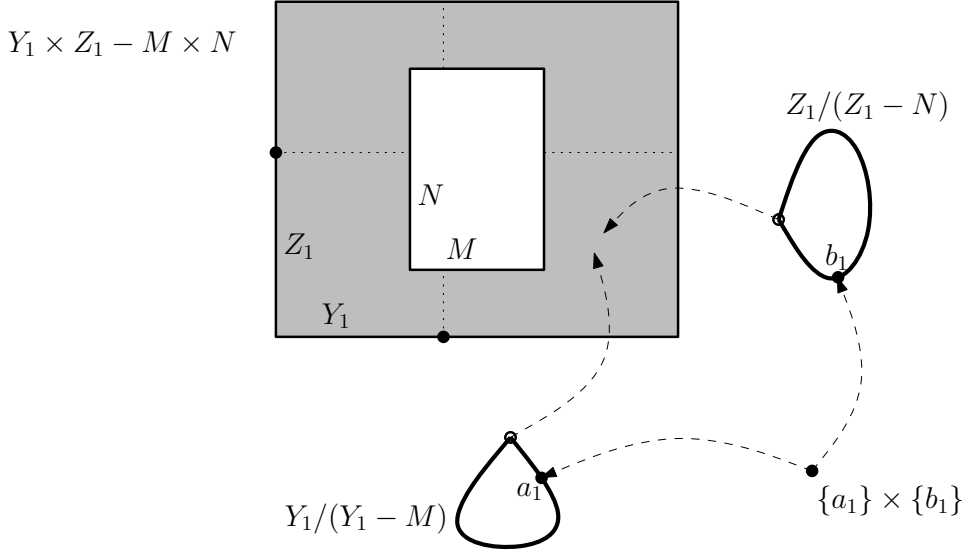


FIGURE 4. Modifying further $Y' \times Z' - M \times N$. (Only a_1 and b_1 showed among a_i 's and b_j 's.)

For $i = 1, 2$, choose a path γ^i connecting b^0 with a vertex $b^i \in B_{b_i} - N'$. Let $\psi^i: S^{n-1} \times [0, 1] \rightarrow (Y_1 - M') \times Z$ be a map constructed by use of γ^i , that is, $\psi^i(x, t) = (\psi(x), \gamma^i(t))$. Combining ψ^1, ψ^2 and the contractions of the spheres $\psi^i(\cdot, 1): S^{n-1} \rightarrow (Y_1 - M) \times \{b^i\}$ within $Y_1 \times \{b^i\}$ for $i = 1, 2$, we obtain the sphere $\varphi: S^n \rightarrow Y \times Z - M' \times N'$. The equator of this sphere is mapped by ψ to $(Y_1 - M') \times \{b^0\}$. By the homotopy equivalences from (2), (3), (4), (5), (6), and (7) we have that φ (seen as a map to $Y \times Z - M \times N$) is homotopically equivalent to a map

$$(8) \quad \varphi': S^n \rightarrow \widehat{Y}_1 \vee \widehat{Y}_2 \subset (\widehat{Y}_1 \vee \widehat{Y}_2) \vee U,$$

mapping one hemisphere to \widehat{Y}_1 , the other to \widehat{Y}_2 , and the equator to the wedge point.

Since Y_1 is contractible, the homotopy pair exact sequence for $(Y_1, Y_1 - M)$ gives us the following isomorphism

$$0 = \pi_n(Y_1) \rightarrow \pi_n(Y_1, Y_1 - M) \xrightarrow{\cong} \pi_{n-1}(Y_1 - M) \rightarrow \pi_{n-1}(Y_1) = 0.$$

Therefore, the nontrivial class $[\psi] \in \pi_{n-1}(Y_1 - M)$ corresponds to the class of $\varphi|: (B^n, S^{n-1}) \rightarrow (Y_1 \times \{b^i\}, (Y_1 - M) \times \{b^i\})$ in $\pi_n(Y_1, Y_1 - M)$, for $i = 1, 2$. Since Y_1 is contractible, and $Y_1 - M$ is path-connected we have (see e.g. [Hat02, Proposition 4.28]) that the homomorphism $\pi_n(Y_1, Y_1 - M) \rightarrow \pi_n(Y_1/(Y_1 - M))$ induced by the quotient map is an isomorphism. Therefore, the map φ' from (8) is homotopically non-trivial. Since the inclusion $\widehat{Y}_1 \vee \widehat{Y}_2 \hookrightarrow (\widehat{Y}_1 \vee \widehat{Y}_2) \vee U$ induces an injection on π_n , we have that $\varphi: S^n \rightarrow Y \times Z - M \times N$ is homotopically nontrivial.

Because C was chosen arbitrarily, it follows that $\pi_n^\infty(Y \times Z)$ is nontrivial. Since $Y \times Z$ is properly $(n+1)$ -equivalent to X , we have that $\pi_n^\infty(X) \neq 0$ and thus, by Proposition 3, $\pi_n^\infty(G) \neq 0$ – contradiction. This concludes the proof of Theorem 2. \square

4. PROOF OF THEOREM 1

In this section we prove Theorem 1 and Corollary 1 from Introduction.

Proof of Theorem 1: Assume that K is not of finite index. Then we have an exact sequence $K \twoheadrightarrow G \twoheadrightarrow Q$ of infinite groups. A simply connected SimplicHatic complex X on which G acts geometrically is AHA by [OŚ13, Corollary 3.2]. Thus $\pi_2^\infty(G) = 0$, by [OŚ13, Theorem 7.4]. Since X is also finite dimensional and contractible (by Whitehead's theorem), by [OŚ13, Theorem 7.6], we have that G has type F_∞ and $\pi_1^\infty(G) \neq 0$. As a subgroup of an AHA group K is itself AHA, and thus of type F_∞ as well, by [OŚ13, Theorem 6.1]. Therefore, the assumptions of Theorem 2 (for the case $n = 2$) are fulfilled. As in the proof of Theorem 2 (see Subsection 3.1) we have that K is the fundamental group of a graph of groups \mathcal{K} with vertex groups $\{K_i\}$ having at most one end each, and with finite edge groups. We have to show that none of K_i has one end – then K is the fundamental group of a graph of finite groups, that is, K is virtually free.

Note that a full subcomplex or a covering of a SimplicHatic complex is again SimplicHatic. Therefore, by a theorem of Hanlon-Martínez-Pedroza [HMP14, Theorem 1.1], each group K_i acts geometrically on a simply connected SimplicHatic complex. If K_i is one-ended then, again by [OŚ13, Theorem 7.6], we have $\pi_1^\infty(K_i) \neq 0$ and thus $\pi_1^\infty(K) \neq 0$ – contradiction. Hence, all K_i are finite, and the proof is completed. \square

Proof of Corollary 1: It is clear that a subcomplex of a nonpositively curved (that is, locally CAT(0)) 2-complex is itself nonpositively curved and two-dimensional, and hence aspherical. Thus some its simplicial subdivision is SimplicHatic. The SimplicHatic property for weakly systolic complexes with SD_2^* links follows from [Osa13b, Proposition 8.2]. In particular, systolic complexes are SimplicHatic (see also [JS06]). Small cancellation graphical small cancellation complexes are (up to some simplicial subdivision) obviously SimplicHatic – see e.g. [OŚ13, Example 4.4] (or use a theorem of Wise [Wis03, Sections 6 & 12] saying that groups acting geometrically on such complexes act geometrically on systolic complexes). Therefore, Corollary 1 follows directly from Theorem 1. \square

For other examples of SimplicHatic complexes extending the list in Corollary 1 (that is, to which Theorem 1 applies) see e.g. [HMP14, Section 1.1] and [OŚ13, Section 4].

5. PROOFS OF COROLLARIES 2, 3 AND 4

Proof of Corollary 2: Assume that K is not of finite index. As in the proof of Theorem 1 it is enough to show that none of the factors $\{K_i\}$ in the splitting of K is one-ended. A finitely generated subgroup of an AHA group is AHA itself [JS07, Corollary 3.4], thus K and all K_i 's are AHA. Furthermore, since every K_i has finite virtual cohomological dimension (see e.g. [Bro94, Chapter VIII.11]), by [OŚ13, Theorem 7.6], none of them is one-ended. It implies that K is virtually free. \square

Proof of Corollary 3: The kernel K is virtually free by Corollary 2. Note that virtually cyclic subgroups of hyperbolic groups are quasiconvex – see [Gro87, Theorem 8.1.D] and [ABC⁺91, Proposition 3.2]. Furthermore, it is shown in [ABC⁺91] that a normal quasiconvex subgroup of a hyperbolic group is either finite or of finite index. Therefore, a virtually cyclic normal subgroup of a hyperbolic group is either finite or of finite index. It follows that K is virtually non-abelian free.

By the same argument as for K , the quotient Q is the fundamental group of a graph of groups \mathcal{Q} with finite edge groups, and vertex groups $\{Q_i\}$ of at most one end (see the proof of Theorem 2, Subsection 3.1). Again, it is enough to show that none of Q_i is one-ended. Suppose that this is not so – say, Q_1 is one-ended. By [Mos96, Theorem A], the quotient Q , and thus also Q_1 are hyperbolic. Hence, Q_1 acts geometrically on a finite dimensional contractible complex – some Rips complex of Q_1 . By [Osa07, Proposition 2.9] (as in the proof of [Osa07, Theorem 3.2]) there is $n \geq 1$ with $\pi_n^\infty(Q_1) \neq 0$. Therefore $\pi_n^\infty(Q) \neq 0$ – contradiction with Theorem 2 since $\pi_{n+1}^\infty(G) = 0$. \square

Proof of Corollary 4: The proof is nearly the same as the one of Corollary 3. The only point is that instead of the finiteness of the virtual cohomological dimension we use the fact that K_i 's, being SimpHAtic, act geometrically on contractible finite dimensional complexes. \square

6. FINAL REMARKS

In Theorem 1 and Corollaries 2, 3 & 4 we use some asphericity conditions in order to apply Theorem 2. However, such assumptions could be weakened by restricting to vanishing only some higher homotopy groups. In particular, there are versions of the statements using the “restricted” versions of AHA or SimpHAticity. One could use e.g. the conditions on filling radius constant for some spherical cycles, like the $S^k FRC$ condition from [JS07, Section 5]. We do not elaborate those issues not seeing any immediate applications. For other related comments and questions see e.g. [JS07, OŚ13].

APPENDIX: TOPOLOGICAL 2-DIMENSIONAL QUASI-HELLY PROPERTY OF
 SYSTOLIC COMPLEXES

In this Appendix we present yet another asphericity-like property of systolic complexes. Besides various asymptotic asphericity features used extensively through the article – like e.g. AHA or vanishing of higher homotopy pro-groups at infinity [JS07, Osa07, Osa08, OŚ13] – the new property reflects the ‘two-dimensional-like’ nature of systolic complexes, implying the normal subgroup theorem studied above. The result – Theorem A below – is a Helly-type theorem, and we believe that it will be of wide use for studying systolic (and, in general, SimpHatic) complexes and groups.³

Recall the Helly property of the Euclidean plane: If, for a given family $\{A_0, A_1, A_3, A_4\}$ of convex subsets, all intersections $A_i \cap A_j \cap A_k$ are nonempty, then $A_0 \cap A_1 \cap A_3 \cap A_4 \neq \emptyset$. In [Świ14] a quasification of such property is proved for systolic complexes: It is shown there that the intersection of neighborhoods of corresponding convex subcomplexes has to be non-empty. Our Theorem A below extends this result: We do not assume that the subcomplexes A_i are convex – we make assumptions on homotopy types of them and of their intersections (thus we call our version “topological”). This is a significant difference: The product (with the product metric) of two complexes satisfying the “convex” version has the same “convex” Helly property, while for our “topological” version the corresponding “Helly dimension” may increase by taking products. Note also that the topological 2-dimensional quasi-Helly property presented below is closely related to the δ -thin tetrahedra property announced in [Els09, Introduction] (there, only geodesic triangles are considered).

Finally, let us note that Theorem A below holds also for a more general class of (SimpHatic) complexes X satisfying the following condition:

(*) Any simplicial map $S \rightarrow X$ from a triangulation S of the 2-sphere can be extended to a simplicial map $D \rightarrow X$ from a triangulation D of the 3-disc, such that the boundary of D is S and D has no new (internal) vertices, that is $D^{(0)} = S^{(0)}$.

Theorem A. *Let A_0, A_1, A_2, A_3 be simply connected subcomplexes of a systolic complex X . Assume that for all $i, j, k \in \{0, 1, 2, 3\}$, the intersections $A_i \cap A_j$ are connected, and $A_i \cap A_j \cap A_k \neq \emptyset$. Then there exists a simplex $\langle v_0, v_1, v_2, v_3 \rangle$ such that $v_i \in A_i$.*

Proof. If $A_1 \cap A_2 \cap A_3 \cap A_4 \neq \emptyset$ then the assertion is trivial. Therefore further we assume that the intersection is empty.

First, we will construct a simplicial 2-sphere, $\varphi: S \rightarrow X$. For all $l \in \{0, 1, 2, 3\}$, pick a vertex $z_l \in A_i \cap A_j \cap A_k$, where $\{i, j, k, l\} = \{0, 1, 2, 3\}$. For all l, k , let $\gamma_{kl} \subset A_i \cap A_j$, where $\{i, j, k, l\} = \{0, 1, 2, 3\}$, be a path

³In fact, our initial – not successful, eventually – approach to the normal subgroups results described in the current article, was based on Theorem A.

connecting z_k and z_l . For $\{i, j, k, l\} = \{0, 1, 2, 3\}$, the cycle consisting of γ_{kl} , γ_{lj} , and γ_{jk} is contained in A_i . Therefore, there is a simplicial filling $\varphi_i: D_i^2 \rightarrow A_i$ of this cycle inside A_i , where D_i^2 is a triangulation of the 2-disc. Combining such fillings, we get a simplicial 2-sphere $\varphi: S \rightarrow X$.

The sphere S has a natural structure of the boundary of the tetrahedron: Vertices are some pre-images \tilde{z}_i of z_i 's; edges are some pre-images $\tilde{\gamma}_{ij}$ of γ_{ij} 's; triangles are the discs D_i 's. However, for our purposes we consider a “dual” structure defined as follows (see Figure 5). For every i , pick a

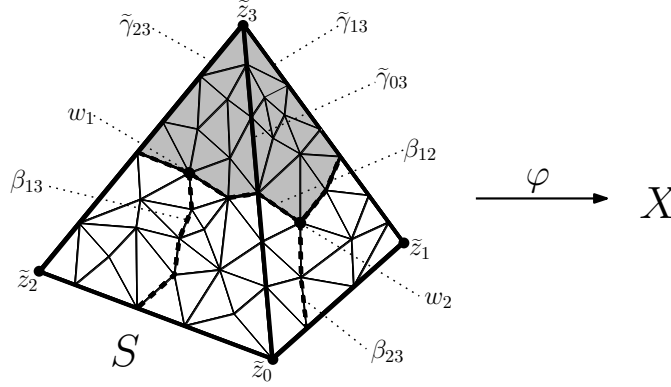


FIGURE 5. The simplicial sphere $\varphi: S \rightarrow X$. The dual triangle $w_0w_1w_2$ is shaded.

vertex w_i inside the interior of D_i – subdivide D_i if necessary (and redefine φ appropriately). For every i, j , find a simple path β_{ij} in S connecting w_i and w_j and contained in $D_i \cup D_j$. Subject to subdividing S , we may assume that the intersection $\beta_{ij} \cap \beta_{kl}$ is either empty or one end-vertex. This defines a new structure of the boundary of the tetrahedron: Vertices are w_i 's; edges are β_{ij} 's; triangles are connected components of the rest. By [JŚ07] (see [Els09, Theorem 2.4]), there exists a simplicial extension $\Phi: D \rightarrow X$ of φ with the following properties: D is a triangulation of the 3-disc, with S being a triangulation of the boundary of D , and without new (that is, internal) vertices.⁴ Now we define a *coloring* of vertices of D , that is a map $c: D^{(0)} \rightarrow \{0, 1, 2, 3\}$:

- $c(w_i) := i$;
- $c(v) \in \{i, j\}$, for $v \in \beta_{ij}$, such that $\varphi(v) \in A_{c(v)}$;
- for a vertex v in the triangle $w_iw_jw_k$, we set $c(v) \in \{i, j, k\}$, such that $\varphi(v) \in A_{c(v)}$.

Observe that the coloring is well-defined (though not unique), and that it is Sperner's coloring of the triangulation of the tetrahedron with vertices w_0, w_1, w_2, w_3 . Therefore, by Sperner's Lemma [Spe28], there exists a 3-simplex in D with vertices v'_0, v'_1, v'_2, v'_3 such that $c(v'_i) = i$, hence $v_i :=$

⁴That is, the condition (*) above is satisfied.

$\Phi(v'_i) \in A_i$, for all $i \in \{0, 1, 2, 3\}$. Since Φ is simplicial, it follows that v_0, v_1, v_2, v_3 are contained in a simplex, and thus the proof is finished. \square

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